THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 3

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1. Prove the L'Hospital's Rule in the case $\frac{\infty}{\infty}$.

2. Suppose $f : (a, b) \to \mathbb{R}$ is everywhere smooth.

   (a) State the Taylor's theorem.

   (b) Using Taylor's theorem, show again that

   $$\lim_{t \to 0} \frac{f(x+t) + f(x-t) - 2f(x)}{t^2} = f''(x)$$

   (c) Show again that $\cos(x) > 1 - \frac{x^2}{2}$ for all $x > 0$.

   (d) Consider the function e^x. Show that for each real number x, we have:

   $$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x$$

   The above is an example of a Taylor series. We will return to this at the end of the term.

3. (Convexity) Let $f : (a, b) \to \mathbb{R}$. To make things simple we assume f is smooth.

   (a) State the definition of f being convex on (a, b).

   (b) Suppose f is smooth. Then f is convex on (a, b) if and only if $f''(x) \geq 0$ on (a, b).

   (c) Fix $c \in (a, b)$. Consider the function $g : (c, b) \to \mathbb{R}$:

   $$g(x) := \frac{f(x) - f(c)}{x - c}$$

   ~~Show that f is convex if and only if the above function g is increasing.~~

   Show that if f is convex, then g is increasing. (Can you come up with a counterexample to the converse?)

   The following is true instead:

   Let $c < x < y < b$. Then:

   $$\frac{f(x) - f(c)}{x - c} \leq \frac{f(y) - f(x)}{y - x}$$

   See if you can prove it.

(d) (Optional) Let $f : (a, b) \to \mathbb{R}$ satisfy the "midpoint convexity": for all $a < x < y < b$,

$$f(\frac{x+y}{2}) \leq \frac{f(x) + f(y)}{2}$$

Show that if f is continuous, then a midpoint convex function is convex.

4. (a) Show that the function $f(x) := x^p$ for $x \geq 0$ is convex if $p \geq 1$.

   (b) Show again that we have the following inequality: For $x, y > 0$, $p \geq 1$,

$$(x + y)^p \leq 2^{p-1}(x^p + y^p)$$

   (c) Show by convexity that Young's inequality is true:

   Let a, b be two positive numbers and $p, q > 1$ be two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

   Hint: Consider the function $\phi(x) := e^x$. Show that ϕ is convex, and let $\lambda := \frac{1}{p}$, whence $1 - \lambda = \frac{1}{q}$.